Int. J. Heat Mass Transfer. Vol. 18, pp 537–551. Pergamon Press 1975. Printed in Great Britain

BUBBLE INDUCED HEAT TRANSFER IN TWO PHASE GAS–LIQUID FLOW

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(*Received* 29 *July* 1974)

Abstract—The influence of gas bubbles on heat transfer in two phase gas–liquid systems has been investigated. Platinum wires have been used as heat-transfer probes and the two phase flow has been simulated by generating a single continuous stream of discrete gas bubbles into a stationary liquid. The contribution of various modes of heat transfer has been determined. It has been found that transient conduction into the liquid is the predominant mode of the bubble induced heat transfer and is responsible for about 75 per cent of heat transfer. Convection contributes the remainder. A theoretical model of the bubble induced heat transfer based on the surface renewal and penetration theory has been developed.

NOMENCLATURE

a, wire radius [m];
C, constant defined by equation (26);
f, frequency of bubble generation [1/s];
Fo, $(\kappa_L t/a^2)$, instantaneous Fourier number;
Fo_m, (κ_L/fa^2), time–mean Fourier number;
k, thermal conductivity [W/mK];
k_0, k_L/k_W, dimensionless thermal conductivity;
L, wire halflength [m];
Nu, Qa^2/Tk_L, theoretically calculated Nusselt number;
Nu_X, $Qa^2/\Delta Tk_L$, experimentally obtained Nusselt number;
Q, heat generation per unit volume of the wire [W/m^3];
r, θ, z, cylindrical space coordinates;
s, $(2n+1)\pi/2$, parameter;
t, time [s];
T, space-average wire temperature [K];
T_F, $(T_m+T_\infty)/2$, film temperature [K];
T_L, temperature of the liquid [K];
T_{L1}, first approximation to the liquid temperature [K];
T_{L2}, liquid temperature due to non-zero initial wire temperature [K];
T_0, initial wire temperature [K];
T_{0A}, initial space-average wire temperature [K];
T_W, temperature of the wire [K];
T_{W1}, first approximation to the wire temperature [K];
T_{W2}, wire temperature due to non-zero initial wire temperature [K];
T_X, experimentally obtained average wire temperature [K];
T_∞, pool temperature [K];
ΔT, $T_X - T_\infty$, temperature difference [K];
y, sa/L, parameter.

Greek symbols

κ, thermal diffusivity [m^2/s];
κ_0, κ_L/κ_W, dimensionless thermal diffusivity;
τ, $1/f$, periodic time [s];
τ_b, bubble residence time on the wire surface [s];
ϕ, function defined by equation (24);
ψ, function defined by equation (25).

Subscripts

1, first approximation to;
2, due to the non-zero initial wire temperature;
i, instantaneous;
L, referring to the liquid;
m, time-mean;
W, referring to the wire.

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1. INTRODUCTION

IT HAS been observed that two phase gas–liquid flow is usually associated with an intense increase in transport rates as compared with a single phase flow under similar flow rate conditions. Heat transfer is one of the transport processes in which this phenomenon is observed.

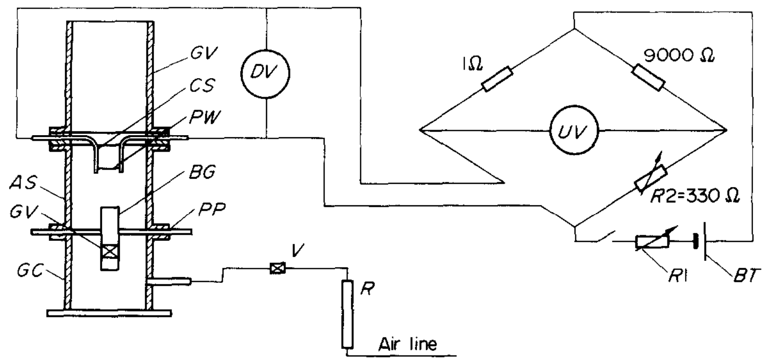

FIG. 1. Schematic diagram of the experimental apparatus.

Although there is some evidence that under certain conditions the latent heat transport is not negligible [1], most authors agree that the enhancement of heat transfer is hydrodynamic in nature and that latent heat transport is unimportant as a heat-transfer mechanism [2–4]. This is demonstrated by the fact that heat-transfer rates in nucleate boiling and in two phase flow without the change of phase (i.e. without evaporation) are of the same order of magnitudes. Hence a theoretical investigation of heat transfer in two phase flow systems in which there is no evaporation, and which is simpler, may aid in understanding of heat-transfer mechanisms in both cases [5].

The object of the reported work is to investigate the precise nature of the heat-transfer mechanism in two phase gas–liquid bubbling flow without evaporation and to determine the contribution of various modes of heat transfer.

Even gas–liquid flow which is characterized by a continuous liquid phase and by a discrete phase of individual gas bubbles is an extremely complicated system for a theoretical investigation [6]. Thus it was decided to simulate it by generating a single continuous stream of discrete gas bubbles into a stationary liquid. The bubble volumes and the frequency of their generation were known. (The mixture of liquid and gas bubbles within the containing vessel will be referred to as the "pool" throughout this paper.)

To compare various mechanisms of heat transfer and their contribution, a heat-transfer probe which can be used to discriminate between conductive and convective modes of heat transfer has been developed [7].

A theoretical model based on the surface renewal and penetration theory and transient conduction as the only mechanism of heat transfer has been developed and has been found to predict the bubble induced heat transfer to a good degree of accuracy.

2. APPARATUS, INSTRUMENTATION AND EXPERIMENTAL TECHNIQUE

2.1 *Experimental apparatus*

The line diagram of the experimental equipment is shown in Fig. 1. The gas is supplied through a system of valves (*V*) and rotameters (*R*), which are used to measure the gas flow rate. The liquid is contained in a glass vessel (*GV*), which is separated from the plenum chamber (*GC*) by a "perspex" plate, in the middle of which is situated the bubble generator (*BG*). The bubble generator consists of a thin stainless steel tube with a valve (*GV*) at one end. Three different bubble generator diameters were used—0·8 mm, 1·6 mm and 2·8 mm. The maximum frequency of bubble generation was obtained with the smallest diameter generator and was of the order of 40 bubbles per second.

The heat-transfer probe, described below, is placed above the centre of the generating orifice. Using additional glass sections (*AS*) it is possible to adjust the distance between the heat-transfer probe and the generating orifice.

2.2 *Heat-transfer probe and its instrumentation*

The development and the design of the probe are described in more detail elsewhere [7]. The schematic diagram of the probe is shown in Fig. 2. A thin platinum wire (*PW*) (serving as the heat-transfer probe) is stretched between two flat copper supports (*CS*), the wire being electrically heated and the copper supports serving as electrical leads of negligible resistance. The length of the wire is equal (or about equal) to the diameter of generated gas bubbles and is set at 4·45 mm.

The platinum wire has three basic functions: (i) it serves as a heat-transfer probe; (ii) using the principles of anemometry it is employed to measure its own instantaneous temperature; and (iii) it is used to measure the frequency of bubble generation.

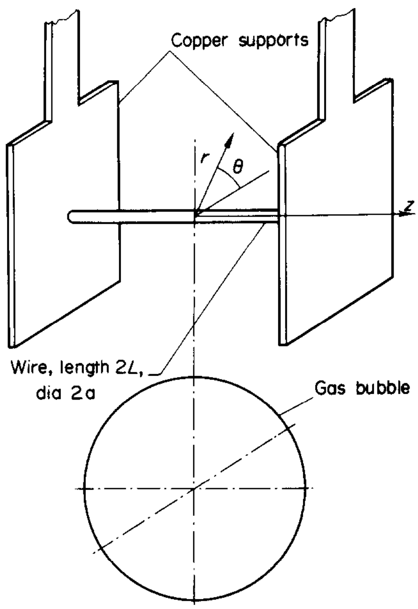

FIG. 2. Schematic diagram of the heat-transfer probe and a gas bubble.

To obtain experimental data from the probe, the wire has been incorporated into one branch of a standard Wheatstone bridge—Fig. 1. The current through the bridge is supplied by a 12 V battery and regulated by the variable resistor $R1$. The digital voltmeter (DV) is used to measure the voltage across the wire and the UV-recorder (UV) to record voltage fluctuations across the bridge. The bridge is balanced by the variable resistor $R2$.

2.3 *Experimental technique*

Experimental technique is described only briefly; more detailed description being given in [7].

The frequency of bubble generation and their size depend on the diameter of the generating orifice, the gas flow rate through this orifice and the properties of the liquid phase. The adjustment of the frequency was effected by adjusting the opening of the valve (GV) and by adjusting the gas back pressure. Volumes of the generated bubbles were determined from the known gas flow rate and the known frequency of bubble generation. Since the bubbles were small it was assumed that they were of a spherical shape and their diameters were calculated from their volumes.

The distance between the wire and the generating orifice was chosen as 10 mm. When this distance was too small, the currents in the liquid caused by bubbles growing at the orifice (local source effect in the liquid) tended to distort the temperature field near the wire. When this distance was too large, the rising bubbles sometimes by-passed the wire because of the oscillatory character of the horizontal component of their velocity. It was found experimentally that when this distance was about 10 mm the distortions of the temperature field were very small and that the bubbles then always hit the wire and were nearly always bisected by it instead of rolling over the wire surface or completely missing it.

The wire was calibrated in liquid at known temperatures and the relationship (the calibration curve) between the wire temperature and the resistance of $R2$ was obtained.

To investigate the mechanism of the bubble induced heat transfer the procedure was as follows. For a given frequency of bubble generation, the time–mean average wire temperature was set approximately at the required level by setting the resistor $R2$ at a value corresponding to this temperature and by adjusting the current through the bridge in such a way that the bridge was approximately in balance. Once the bridge was roughly balanced, the UV-recorder traces were obtained. Using calculated calibration constants for the UV-recorder traces [7], the variations of the instantaneous wire temperature with time were obtained. The exact time–mean average wire temperature was then calculated over the whole cycle (from point O to point O'—Fig. 12) by graphical integration.

The frequency of bubble generation was then determined from the frequency of the periodic fluctuations of the instantaneous wire temperature.

The rate of heat generation per unit volume of the wire, Q, was calculated from the voltage across the wire and the average wire resistance corresponding to the time–mean average wire temperature. (The change of the average wire resistance and the voltage with time was neglected. The error introduced by these two approximations was small and was estimated to be below 3 per cent.)

The experimental instantaneous Nusselt number, $(Nu_X)_i$, corresponding to a particular value of the instantaneous Fourier number, was then calculated from the known values of the rate of heat generation per unit volume of the wire, Q, and the instantaneous average temperature difference $\Delta T_i = (T_X)_i - T_\infty$. Similarly the experimental time–mean Nusselt number (calculated over the whole cycle) was obtained for each particular value of the time–mean Fourier number.

3. EXPERIMENTAL RESULTS

Three different liquids were used for the experimental investigation of the phenomenon. Their properties calculated at the film temperature $T_F = 25°C$ are shown in Table 1. Air was used as the gas phase. Probe wires of four different diameters were used ranging from 41·15

Table 1. Liquids used for the experimental investigation

Liquid	Thermal conductivity (W/mK)	Specific heat (J/kgK)	Density (kg/m^3)	Viscosity (kg/ms)
Water	0·615	4180	1000	0·000895
n-heptane	0·140	2130	700	0·00042
50 % aqueous solution of glycerol	0·420	3390	1125	0·0057

to 305 μm. The liquid temperature was kept at 20°C and unless stated otherwise, the time-mean temperature difference was set at about 10°C (ΔT_m = 10°C).

The variation of the instantaneous average wire temperature with time is shown in Fig. 3. This figure was obtained by photographing some of the UV-recorder traces. The ripples on these traces are due to noise. These temperature variations were obtained using a platinum wire 41·15 μm diameter in water. The choice of the liquid in the pool had little effect on the general character of the temperature curves, but the influence of the wire diameter was profound. The greatest sensitivity to bubble frequency, and hence the clearest outputs, was obtained with the smallest diameter wire, since its heat capacity was small and the distortions due to it were minimized.

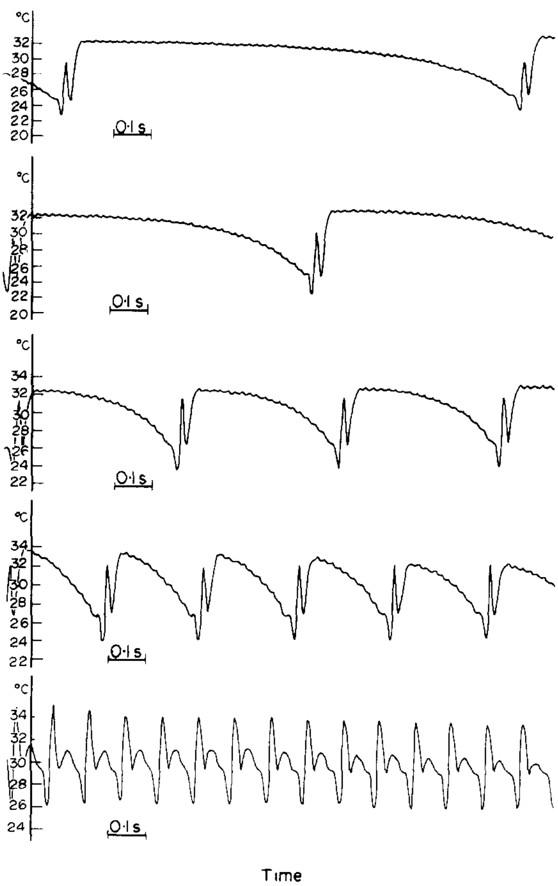

FIG. 3. Variation of the instantaneous wire temperature with time (water, a = 20·6 µm, A = 108, T_∞ = 20°C). From top to bottom: f = 0·8, 1·2, 2·3, 3·8 and 10·4 l/s.

Some experimentally obtained instantaneous Nusselt numbers are plotted against the instantaneous Fourier numbers in Figs. 4 and 5. The discrete experimental points were obtained from the rising parts of the temperature curves (point 1 to point 2—Fig. 12). For each cycle the rising part of the temperature curve was divided into six equal parts and hence seven experimental points were obtained from each cycle; since more cycles were used there is some inevitable scatter in these figures.

Some of the experimental time-mean Nusselt numbers are plotted against the time-mean Fourier numbers in Figs. 6–10.

Experiments were conducted to investigate the influence of elevated temperatures on the mechanism of the bubble induced heat transfer. All experimental conditions except the film temperature, T_F, remained the same as above. Water was used as the liquid phase. Experimental results are shown in Fig. 11.

4. THEORETICAL ANALYSIS

4.1 *Assumptions used in the theoretical model*

A theoretical model of the bubble induced heat transfer in the present system, based on the surface renewal and penetration theory [8, 9], has been developed. The model is based on the following assumptions:

(i) The heat-transfer process is periodic with period $\tau = 1/f$, where f is the frequency of bubble generation.

(ii) The presence of gas bubbles in the pool is responsible for the continuous mixing of the liquid in the pool. Because of the continual mixing in the bulk of the pool, the temperature of the liquid throughout

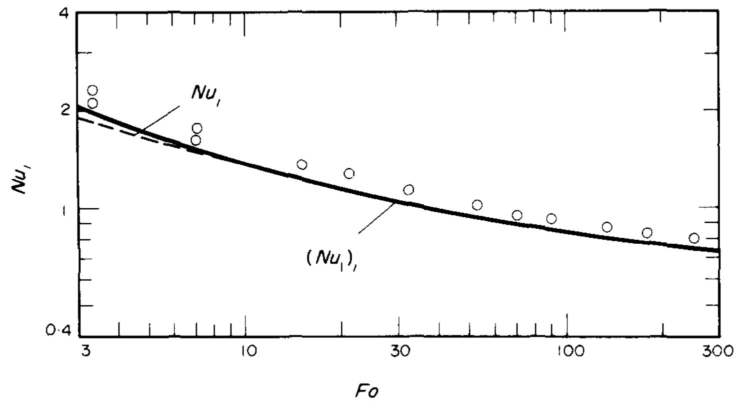

FIG. 4. Instantaneous heat transfer from a 41·15 μm dia wire into water ($A = 108$, $T_F = 25°C$).

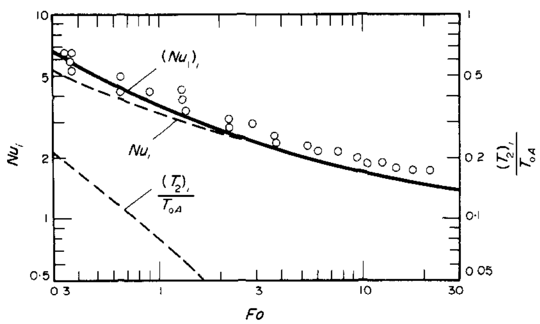

FIG. 5. Instantaneous heat transfer from a 125 μm dia wire into water ($A = 35·56$, $T_F = 25°C$).

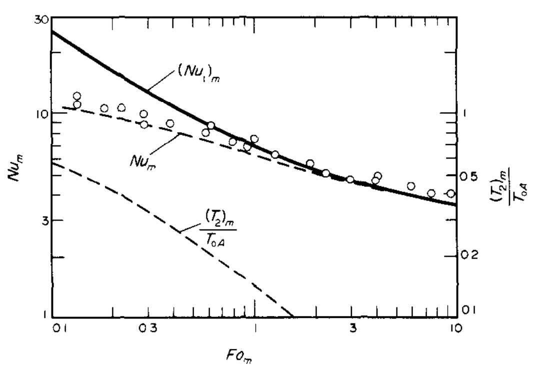

FIG. 6. Time-mean heat transfer from a 305 μm dia wire into water ($A = 14·6$, $T_F = 25°C$).

the bulk of the pool remains constant. When a bubble passes the surface of the heat-transfer probe, the "old" liquid, which has been heated by the wire prior to the bubble arrival is replaced by "fresh" liquid from the bulk of the pool, which was brought by the passing bubble in its wake. While this liquid is in contact with the surface of the heat-transfer probe, heat is transferred to it by a non-steady heat-transfer process. For a particular case, the amount of heat transferred to it depends on the duration of the contact time between the surface of the heat-transfer probe and the liquid. After a certain time, known as the "residence time" of the liquid on the surface of the heat-transfer probe, the next bubble arrives and the now "old" liquid is again replaced by the fresh liquid from the bulk of the pool. In this way the liquid on the surface of the heat-transfer probe is being replaced periodically at frequency f.

(iii) The diameter of passing bubbles is approximately equal to the length of the probe. Hence it is assumed that the liquid is being effectively replaced over the whole surface area of the heat-transfer probe. It is further assumed that the residence time of gas bubbles on the surface of the heat-transfer probe is negligible compared with the residence time of the liquid there. This implies that only liquid is in contact with the surface of the heat-transfer probe and hence that only liquid is responsible for the heat transfer from the probe to the pool.

(iv) It is further assumed that all liquid and wire material properties remain constant, that dissipation of mechanical energy can be neglected and that only resistance heating of the wire is considered.

In order to define the model completely one must know the boundary conditions of the system. These generally depend on the heat-transfer probe used. As a result of the construction of the present probe the following assumptions about its behaviour are made:

(i) Since the electrical and thermal resistances of the copper supports (Fig. 2) are small compared with those of the wire material and their surface area is large compared with that of the wire, it is assumed that they remain at the pool temperature, T_∞, throughout the process.

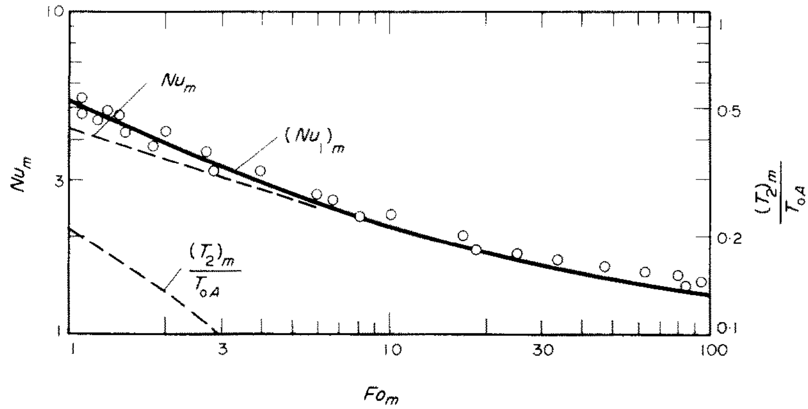

FIG. 7. Time-mean heat transfer from a 125 μm dia wire into water ($A = 35{\cdot}6$, $T_F = 25°C$).

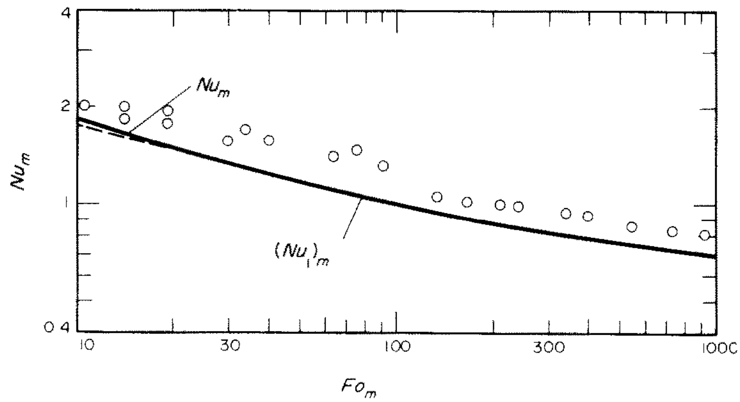

FIG. 8. Time-mean heat transfer from a 41·15 μm dia wire into water ($A = 108$, $T_F = 25°C$).

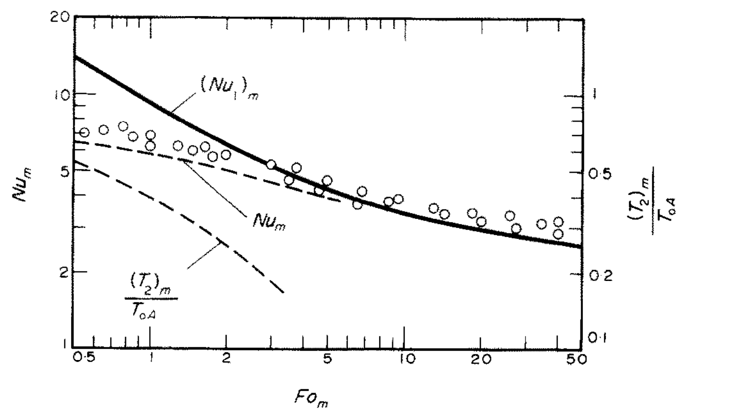

FIG. 9. Time-mean heat transfer from a 125 μm dia wire into *n*-heptane ($A = 35{\cdot}6$, $T_F = 25°C$).

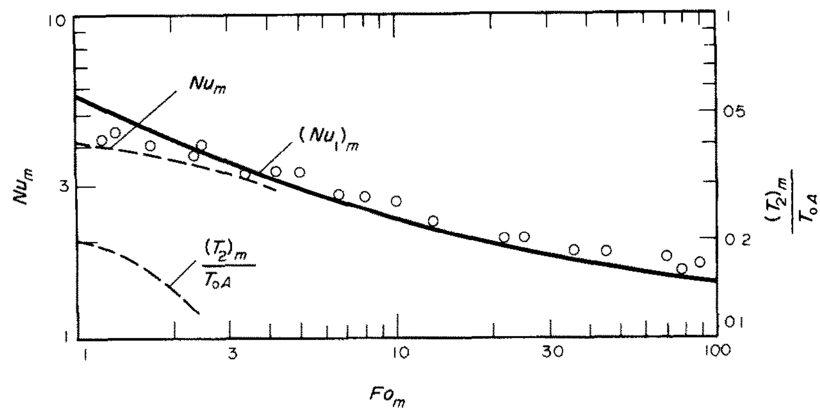

FIG. 10. Time-mean heat transfer from a 125 µm dia wire into 50/50 glycerol solution ($A = 35{\cdot}6$, $T_F = 25°C$).

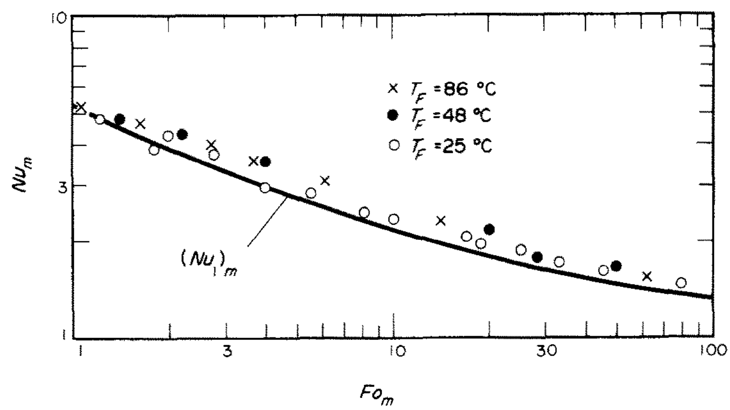

FIG. 11. Time-mean heat transfer from a 125 µm dia wire into water at elevated temperatures ($A = 35{\cdot}6$).

(ii) Finally it is assumed that the radial extent of the copper supports and of the liquid between them is infinite, compared with the very small radius of the wire. Hence the system can be regarded as axisymmetric about the z-axis.

In order to make the mathematical description of the system simpler, the pool temperature is set at the reference zero ($T_\infty = 0$). Only the slab of the liquid between the two copper supports is considered and it is assumed that each heating period starts at the moment when the liquid on the wire surface has just been renewed. Using the above assumptions the full energy description of the process can be obtained. The energy equations together with momentum and continuity equations provide the complete description of the system. Thus, at least in principle, the solution could be obtained.

4.2 *Non-steady heat-transfer mechanism*

The system of differential equations describing the bubble induced heat transfer in the present system [7] is very complex and to solve it, even numerically, would be extremely difficult. So an alternative method of attack must be developed. The method is based on separating conductive and convective modes of heat transfer.

While the wire is in contact with the surrounding liquid, the heat is transferred from the wire to the liquid by a non-steady heat-transfer process. In the present system this non-steady heat-transfer mechanism is defined generally as consisting of a conductive and a convective component. The convective component of the heat-transfer mechanism can be calculated only after the velocity field in the wire vicinity has been determined. The calculation of the velocity field in the wire vicinity causes the main difficulties in finding a full analytical solution. As mentioned previously, the present probe can be used to discriminate between conductive and convective components of heat transfer [7]. This is due to the fact that non-steady heat transfer from the probe to the surrounding liquid, when conduction is the only mode of heat transfer, can be calculated theoretically to a high degree of accuracy. Hence the effect of convection can be determined by comparing the experimentally observed heat transfer from the probe under mixed mode conditions (conduction and convection) with that one determined theoretically, in which conduction is the only mode of heat transfer. Thus in the subsequent analysis the effect of additional convection is neglected and it is assumed that conduction is the only mode of the bubble induced heat transfer.

The neglect of the convective mode of heat transfer implies the following simplifying assumption: Gas bubbles cause the renewal of the liquid on the wire surface only, and the liquid remains stationary once in contact with the surface of the probe and there are no convection currents within the liquid. This is clearly the most controversial assumption and its validity will be discussed later. If the experimental heat-transfer rates are higher than the theoretical ones, which assume that conduction is the only mechanism of the bubble induced heat transfer, the assumption does not hold. In that case convection provides an important contribution to the bubble induced heat transfer and its importance can be assessed from the difference between experimental and theoretical results (the greater the relative difference, the greater the contribution of convection).

4.3 *Influence of the initial wire temperature*

At the start of each heating cycle, when the liquid has just been replaced on the surface of the heat-transfer probe, the initial temperature of the liquid in the wire vicinity is the same as the temperature of the liquid in the bulk of the pool, T_∞. The initial temperature of the wire is different. Because the probe wire has finite heat capacity, finite times are required for finite changes of the wire temperature. Hence the initial wire temperature is different from the initial temperature of the liquid. The initial wire temperature depends on many factors, such as the bubble residence time on the wire surface, thermophysical properties of the bubble gas, etc. To discuss the influence of the initial wire temperature, T_0, the energy equations are presented in the following way [7]:

Let

$$T_L = T_{L1} + T_{L2} \tag{1}$$

$$T_W = T_{W1} + T_{W2} \tag{2}$$

such that

$$t \geqslant 0,\ r > a,\ |z| \leqslant L$$

$$\frac{\partial T_{L1}}{\partial t} = \kappa_L \left(\frac{\partial^2 T_{L1}}{\partial r^2} + \frac{1}{r}\frac{\partial T_{L1}}{\partial r} + \frac{\partial^2 T_{L1}}{\partial z^2} \right) \tag{3}$$

$$t \geqslant 0,\ r \leqslant a,\ |z| \leqslant L$$

$$\frac{\partial T_{W1}}{\partial t} = \kappa_W \left(\frac{\partial^2 T_{W1}}{\partial r^2} + \frac{1}{r}\frac{\partial T_{W1}}{\partial r} + \frac{\partial^2 T_{W1}}{\partial z^2} \right) + Q\frac{\kappa_W}{k_W} \tag{4}$$

subject to

$$t = 0,\ |z| \leqslant L \quad T_{L1} = T_{W1} = 0 \tag{5}$$

$$t \geqslant 0,\ |z| = L \quad T_{L1} = T_{W1} = 0 \tag{6}$$

$$t \geqslant 0,\ z = 0 \quad \frac{\partial T_{L1}}{\partial z} = \frac{\partial T_{W1}}{\partial z} = 0 \tag{7}$$

$$t \geqslant 0,\ r = a,\ |z| \leqslant L \quad T_{L1} = T_{W1} \tag{8}$$

$$-k_L\frac{\partial T_{L1}}{\partial r} = -k_W\frac{\partial T_{W1}}{\partial r} \tag{9}$$

and

$$t \geqslant 0,\ r > a,\ |z| \leqslant L$$

$$\frac{\partial T_{L2}}{\partial t} = \kappa_L \left(\frac{\partial^2 T_{L2}}{\partial r^2} + \frac{1}{r}\frac{\partial T_{L2}}{\partial r} + \frac{\partial^2 T_{L2}}{\partial z^2} \right) \tag{10}$$

$$t \geqslant 0,\ r \leqslant a,\ |z| \leqslant L$$

$$\frac{\partial T_{W2}}{\partial t} = \kappa_W \left(\frac{\partial^2 T_{W2}}{\partial r^2} + \frac{1}{r}\frac{\partial T_{W2}}{\partial r} + \frac{\partial^2 T_{W2}}{\partial z^2} \right) \tag{11}$$

subject to

$$t = 0,\ |z| \leqslant L \quad T_{L2} = 0 \tag{12}$$

$$T_{W2} = T_0(r, z) \tag{13}$$

$$t \geqslant 0,\ |z| = L \quad T_{L2} = T_{W2} = 0 \tag{14}$$

$$t \geqslant 0,\ z = 0 \quad \frac{\partial T_{L2}}{\partial z} = \frac{\partial T_{W2}}{\partial z} = 0 \tag{15}$$

$$t \geqslant 0,\ r = a,\ |z| \leqslant L \quad T_{L2} = T_{W2} \tag{16}$$

$$-k_L\frac{\partial T_{L2}}{\partial r} = -k_W\frac{\partial T_{W2}}{\partial r}. \tag{17}$$

The solution of equations (10)–(17) provides only a transient non-trivial temperature field, due to the non-zero initial wire temperature. The importance of this solution can be summarized by the following statement: The greater the instantaneous Fourier number, Fo, the smaller the influence of the non-zero initial wire temperature on the total temperature field in and around the wire [7]. Hence the solution of equations (3)–(9) will provide the first approximation to the temperature field in and around the wire when conduction is the only mechanism of heat transfer between the wire and the liquid, and its accuracy will increase with increasing values of the instantaneous Fourier number.

This approximation implies the assumption that at the beginning of each heating cycle the temperature of the wire drops instantaneously to the initial temperature of the liquid, T_∞, which is set at the reference zero (Section 4.1).

4.4 *First approximation to the wire temperature*

Equations (3)–(9) provide the first approximation to the temperature field in and around the wire when conduction is the only mode of heat transfer between the probe and the surrounding liquid. They can be solved numerically, but they are still too complex for analytical treatment. Hence one additional simplifying assumption is considered.

Since the wire is very thin and is made of material whose thermal conductivity is very much greater than that of the surrounding liquid, the radial distribution of temperature within the wire will be nearly uniform. The wire then may be for many purposes regarded as a finite rod with heat generated within it and being dissipated (a) from the outer surface by conduction into the surrounding liquid and (b) from its ends by conduction to the copper supports [10]. The differential equation (4) and boundary conditions (8) and (9) then simplify into one boundary condition (22) (Appendix 1).

Hence the original system of equations (3)–(9) can be approximated by the following equations:

$$t \geqslant 0,\ r \geqslant a,\ |z| \leqslant L$$

$$\frac{\partial T_{L1}}{\partial t} = \kappa_L \left(\frac{\partial^2 T_{L1}}{\partial r^2} + \frac{1}{r}\frac{\partial T_{L1}}{\partial r} + \frac{\partial^2 T_{L1}}{\partial z^2} \right) \tag{18}$$

subject to

$$t = 0,\ r \geqslant a,\ |z| \leqslant L \quad T_{L1} = 0 \tag{19}$$

$$t \geqslant 0,\ r \geqslant a,\ |z| = L \quad T_{L1} = 0 \tag{20}$$

$$t \geqslant 0,\ r \geqslant a,\ z = 0 \quad \frac{\partial T_{L1}}{\partial z} = 0 \tag{21}$$

$$t \geqslant 0,\ r = a,\ |z| \leqslant L \quad \frac{1}{\kappa_W}\frac{\partial T_{L1}}{\partial t} = \frac{\partial^2 T_{L1}}{\partial z^2} + \frac{2k_L}{ak_W} \times \frac{\partial T_{L1}}{\partial r} + \frac{Q}{k_W}. \tag{22}$$

Equation (18), subject to equations (19)–(22), was first solved by Jaeger [10], who obtained the solution for the maximum instantaneous wire temperature only. Solution, suitable for the following analysis, method of which is shown briefly in Appendix 2, was obtained by the present author as

$$t \geqslant 0,\ r \geqslant a,\ |z| \leqslant L$$

$$\frac{T_{L1}k_L}{a^2 Q} = \sum_{n=0}^{\infty} \frac{(-1)^n}{s} \cos\frac{sz}{L} \int_0^\infty \left[1 - e^{-(x^2+y^2)Fo}\right] \times \frac{Y_0\left(\frac{r}{a}x\right)\psi - J_0\left(\frac{r}{a}x\right)\phi}{(x^2+y^2)(\phi^2+\psi^2)} x\,dx \tag{23}$$

where

$$\phi = xY_1(x) + CY_0(x) \tag{24}$$

$$\psi = xJ_1(x) + CJ_0(x) \tag{25}$$

$$C = \frac{1}{2k_0}\left[y^2 - (x^2+y^2)\kappa_0\right] \tag{26}$$

and where Bessel functions have their usual meanings.

Temperature of the wire is obtained from equation (23) with $r = a$ (Appendix 1).

In order to compare theoretical and experimental results with as low error as possible, it is desirable to express the theoretical solution in terms of directly measurable parameters. These will be obtained by considering the experimental application of the probe. It is not practicable to detect the local wire temperature; in practice the wire temperature is determined from the electrical resistance of its entire length. If, as in the present case, the wire is used in a similar way to a constant current anemometer, a certain space-average wire temperature will be measured. This space-average wire temperature is the directly measurable parameter upon which the theoretical solution should be based. It is therefore necessary to find the relationship between the local wire temperature distribution, T_W, at any time t and the space-average wire temperature, T, at the same instant. Hereafter the term "average wire temperature" will be used to denote the space-average wire temperature.

Because the wire temperature fluctuations are small, the relationship between the wire temperature and its resistance can be approximated by a linear function. The average wire temperature is then defined as [7]

$$T = \frac{1}{2L}\int_{-L}^{L} T_W\,dz. \tag{27}$$

The instantaneous average wire temperature, $(T_1)_i$, is obtained by substituting $r = a$ into equation (23) to obtain the local wire temperature and then by substituting the local wire temperature T_{W1} into equation (27) to calculate $(T_1)_i$ as

$$\frac{(T_1)_i k_L}{a^2 Q} = \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{s^2}\int_0^\infty \left[1 - e^{-(x^2+y^2)Fo}\right] \times \frac{x\,dx}{(x^2+y^2)(\phi^2+\psi^2)}. \tag{28}$$

Equation (28) gives the instantaneous average wire temperature. From the point of view of the overall time–mean heat transfer, the time–mean average wire temperature if of far greater importance. The time–mean average wire temperature can be calculated only if the residence time distribution function of the elements of the liquid on the surface of the heat-transfer probe is known. It is assumed (Section 4.2) that the liquid on the wire surface is stationary and this implies that the required age distribution function is the Higbie's uniform age distribution function [8] with the mean residence time of the liquid on the wire surface given by the frequency of bubble generation. Hence the first approximation to the time-mean average wire temperature is defined by

$$(T_1)_m = \frac{1}{\tau}\int_0^\tau (T_1)_i \,\mathrm{d}t. \tag{29}$$

Substituting equation (28) into equation (29) the first approximation to the time-mean average wire temperature $(T_1)_m$ is then calculated as

$$\frac{(T_1)_m k_L}{a^2 Q} = \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{s^2}\int_0^\infty \times \left\{1 - \frac{1}{(x^2+y^2)Fo_m}\left[1-\mathrm{e}^{-(x^2+y^2)Fo_m}\right]\right\} \times \frac{x\,\mathrm{d}x}{(x^2+y^2)(\phi^2+\psi^2)}. \tag{30}$$

4.5 *Temperature field due to initial wire temperature*

The temperature field due to the influence of the initial wire temperature is given by the solution of equations (10)–(17). It is again assumed that the radial distribution of the wire temperature is nearly uniform and hence the initial wire temperature can be rewritten as

$$T_0(r,z) = T_{0A} fn(z) \tag{31}$$

where $fn(z)$ is normalized such that

$$\frac{1}{2L}\int_{-L}^{L} fn(z)\,\mathrm{d}z = 1 \tag{32}$$

and hence T_{0A} is the initial average wire temperature.

In order to solve equations (10)–(17) function $fn(z)$ and temperature T_{0A} must be known.

It has been shown experimentally (Fig. 3) that the initial average wire temperature is approximately equal to the experimentally determined time-mean average wire temperature, $(T_X)_m$. Because the theoretical analysis developed in this work is based on the assumption that the pool temperature, T_∞, is zero, the initial average wire temperature which must be used here is given by the difference between the experimentally determined time-mean average wire temperature, $(T_X)_m$, and the pool temperature, T_∞.

$$T_{0A} = (T_X)_m - T_\infty. \tag{33}$$

It is difficult to determine the initial wire temperature distribution and hence the simplest form of the function $fn(z)$, satisfying all boundary conditions, is assumed. Thus a parabolic function is chosen, so that

$$fn(z) = 1{\cdot}5(1-z^2). \tag{34}$$

Equations (10)–(17) can now be solved numerically [7] and the instantaneous and the time-mean average wire temperatures due to the initial wire temperature, T_0, can be calculated. Hence $(T_2)_i$ and $(T_2)_m$ can be obtained. [Equation (27) is used to calculate the average wire temperature and equation (29) the time-mean average wire temperature.]

4.6 *Conduction model of the bubble induced heat transfer*

Full solution for the temperature field in and around the wire is defined by equations (1) and (2). Similarly the instantaneous and the time-mean average wire temperatures can be defined as

$$T_i = (T_1)_i + (T_2)_i \tag{35}$$

$$T_m = (T_1)_m + (T_2)_m \tag{36}$$

respectively. Equations (35) and (36) can be rewritten in dimensionless form as

$$\frac{T_i k_L}{Qa^2} = \frac{(T_1)_i k_L}{Qa^2} + \frac{(T_2)_i}{T_{0A}}\frac{[(T_X)_m - T_\infty]k_L}{Qa^2} \tag{37}$$

$$\frac{T_m k_L}{Qa^2} = \frac{(T_1)_m k_L}{Qa^2} + \frac{(T_2)_m}{T_{0A}}\frac{[(T_X)_m - T_\infty]k_L}{Qa^2}. \tag{38}$$

It can be shown that $Qa/2(T_1)_i$ and $Qa/2(T_1)_m$ can be regarded as first approximations to the instantaneous and the time-mean heat-transfer coefficients for the conduction model of the bubble induced heat transfer in the present system respectively. Similarly $Qa^2/(T_1)_i k_L$ and $Qa^2/(T_1)_m k_L$ can be regarded as first approximations to the instantaneous and the time-mean Nusselt numbers respectively. Furthermore, for the above value of the initial average wire temperature, T_{0A} [equation (33)], the group $Qa^2/[(T_X)_m - T_\infty]k_L$ is equal to the experimentally determined time-mean Nusselt number, $(Nu_X)_m$.

Hence

$$(Nu_1)_i = \frac{Qa^2}{(T_1)_i k_L} \tag{39}$$

$$(Nu_1)_m = \frac{Qa^2}{(T_1)_m k_L} \tag{40}$$

where $(Nu_1)_i$ and $(Nu_1)_m$ are first approximations to the instantaneous and the time-mean Nusselt numbers respectively.

Similarly

$$Nu_i = 1\Bigg/\left[\frac{1}{(Nu_1)_i} + \frac{(T_2)_i}{T_{0A}}\frac{1}{(Nu_X)_m}\right] \tag{41}$$

$$Nu_m = 1\Bigg/\left[\frac{1}{(Nu_1)_m} + \frac{(T_2)_m}{T_{0A}}\frac{1}{(Nu_X)_m}\right] \tag{42}$$

where Nu_i and Nu_m are full solutions for the instantaneous and the time-mean Nusselt numbers respectively.

If then the properties of the liquid and of the wire material and the wire dimensions are known, the Nusselt numbers can be calculated as functions of the Fourier numbers.

The theoretical predictions are compared with the experimental data in Figs. 4–11.

5. DISCUSSION

5.1 *Instantaneous average wire temperature*

A typical instantaneous wire temperature vs time diagram is shown in Fig. 12. No simultaneous photographs of rising gas bubbles were taken. However, a simple explanation of this temperature–time profile can be made.

drop in the heat-transfer rate and hence in turn a sudden rise of the wire temperature. This corresponds to the temperature rise of the wire from point 3 to point 0′. When the bubble leaves the wire, cold pool liquid in its wake hits the wire and the whole process is repeated again.

The residence time of gas bubbles on the wire, τ_b, corresponds to the time interval necessary for a bubble to pass the wire and is of the order of 10–20 ms. This agrees well with the experimental observations of bubble sizes and velocities made during the present investigation. The bubble diameter was about 4·5 mm and the corresponding bubble velocity about 300 mm/s, implying a residence time of the bubble on the wire surface of about 15 ms.

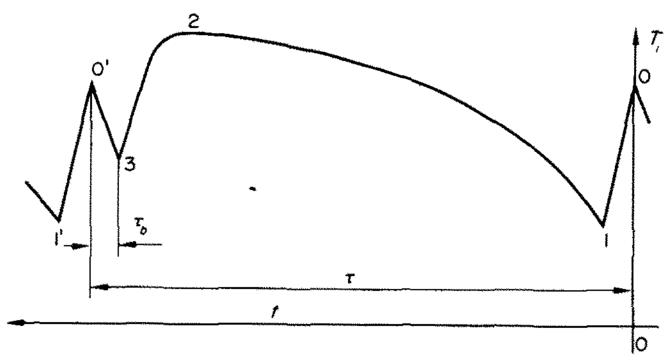

FIG. 12. A typical diagram of the instantaneous average wire temperature as a function of time.

Referring to Fig. 12, assume that the periodic process starts at point 0, when the wake of a passing gas bubble brings fresh liquid in contact with the wire. At that moment two heat-transfer processes are competing. The first is the heating of the wire by the heat generated within it (which is present throughout the process); the second is the simultaneous rapid cooling of the wire by the fresh cool liquid. From point 0 to point 1 the cooling rate is greater than the heating rate, at point 1 they are equal, and from point 1 to point 2 the transient heating of the wire and of the liquid surrounding it is the only heat-transfer process present. The transient heating of the wire and of the surrounding liquid is undisturbed until the next bubble approaches the wire. Then the nose of this bubble forces the "old" warm liquid away and replaces it partially with the fresh liquid, causing some cooling of the wire. This corresponds to the temperature drop from point 2 to point 3. At point 3 the gas bubble itself comes into contact with the wire and at least some of the liquid in the wire vicinity is replaced by the gas contained inside the gas bubble. This causes a marked

Figure 3 demonstrates that the initial average wire temperature, T_{0A} (point 0, Fig. 12), is in most cases approximately equal to the time–mean average wire temperature. This value of the initial average temperature was used to determine temperatures $(T_2)_i$ and $(T_2)_m$—Section 4.5.

5.2 *Surface renewal and other assumptions*

The rate of heating of the wire during the time when a bubble is present on it (corresponding to the time interval from point 3 to point 0′, Fig. 12) was investigated. The observed rate of temperature rise of the wire was far below the value expected for heating of a bare platinum wire in air, even when the time constants of the wire, bridge and recording instruments were allowed for. From this it was deduced that there was a liquid film attached to the wire during the bubble residence on the wire surface.

Next it was investigated if this liquid film remained attached to the wire permanently. The question about the permanent attachment of the liquid film is closely related to the question about the effectiveness of the surface renewal of the liquid by the action of passing bubbles. This was investigated as follows:

Assume that when the wake of a passing bubble hits the wire (point 0, Fig. 12) the liquid on the surface of the wire is completely replaced and that conduction is the only mechanism of the subsequent heat transfer. The instantaneous average wire temperature is then given by equation (35).

Temperature T_i goes through a minimum, the value of which is calculated theoretically from equation (35). This minimum has also been observed experimentally (point 1, Fig. 12 and Fig. 3). The experimental values of the minimum have been found on average to be about 25 per cent lower than the theoretical ones calculated from equation (35) on the basis of the complete surface renewal of the liquid and conduction as the only mechanism of heat transfer. It is shown in Section 5.3 that when Fourier numbers are sufficiently large [i.e. when temperature $(T_2)_i$ is negligible compared with temperature $(T_1)_i$, which can be determined very accurately theoretically], convection is responsible for about 25 per cent of the bubble induced heat transfer. This implies that for identical conditions, experimental temperatures are about 25 per cent lower than the theoretical ones, which are calculated on the assumption that conduction is the only mechanism of the bubble induced heat transfer. It is reasonable to assume that the same situation occurs for extremely small Fourier numbers, which must be considered when investigating the effectiveness of the surface renewal of the liquid. This then necessarily means that the renewal of the liquid on the wire surface is complete and that the liquid does not remain attached to the wire permanently. Had this been the case, the experimentally observed minimum wire temperatures would have been nearer to, or even above, the theoretically calculated minimum wire temperatures.

Hence the liquid film is attached to the wire only during the presence of gas bubbles on it when, perhaps, the momentum of gas bubbles is not sufficiently large to force all liquid away from the wire vicinity. When the bubble passes the wire the large momentum of the bubble wake causes a complete liquid replacement in the wire vicinity.

For the calculation of the time–mean heat-transfer coefficients, it was assumed that the residence time of gas bubbles on the wire surface, τ_b, was negligible and that only bubble wakes were responsible for the surface renewal of the liquid. Clearly this is not the case, because the bubble residence time, τ_b, is finite and the bubble nose is responsible for a partial liquid renewal on the wire surface (Figs. 3 and 12). The influence of these two effects will be more important for higher frequencies of bubble generation when the ratio τ_b/τ is relatively large. Because of the influence of two opposing effects during the liquid renewal, which are most prominent for higher frequencies of bubble generation (Fig. 3), the error due to the above mentioned discrepancies will be attenuated. The two opposing effects are the partial drop of the wire temperature during the approach of the bubble nose (interval 2 to 3, Fig. 12) and the wire temperature rise during the presence on the wire of the bubble itself (interval 3 to 0′, Fig. 12). This is schematically shown in Fig. 12. The broken line represents the case of an ideal liquid renewal and the two above mentioned effects cancel, at least partially, each other out.

5.3 *Bubble induced heat transfer*

It has been shown in the preceding Section that bubble induced liquid replacement on the surface of the probe wire is a 100 per cent effective mechanism of mass transport there. Hence the theoretically derived solution for the instantaneous heat transfer from the wire to the surrounding liquid, based on transient conduction as the only mechanism of heat transfer, should result in heat-transfer coefficients which are lower than those obtained experimentally. This must be the case because the additional convection in any form, which increases the experimentally obtained heat-transfer coefficients, is neglected in the theoretical analysis. Because of the liquid moving with the bubble wake, the effect of forced convection is most prominent for short liquid residence times and hence for small values of the instantaneous Fourier number, Fo. On the other hand, the effect of natural convection becomes important for large liquid residence times (large values of the instantaneous Fourier number). For intermediate values of liquid residence time, both types of convection are important.

Figures 4 and 5 demonstrate that the above requirement is well satisfied. Experimental results are about 25 per cent higher than the theoretical predictions which are based on transient conduction as the only mechanism of the bubble induced heat transfer. This confirms the assumption that transient conduction is the most important mechanism of the bubble induced heat transfer. Transient conduction is responsible for about 75 per cent of the total heat transfer and liquid convection contributes the remainder.

Situations similar to the case of the instantaneous bubble induced heat transfer can be observed on graphs of the time–mean bubble induced heat transfer (Figs. 6–10)—all experimental results lie 20–30 per cent above the theoretical predictions. The experimental results of the time–mean heat transfer, coupled with the experimental observation that transient conduction is responsible for 75 per cent of heat transfer also in the case of the instantaneous heat transfer, confirm indirectly the assumption that the bubble wakes are primarily responsible for the renewal of the liquid on the wire surface and, furthermore, that the effect of the finite residence time of the bubbles on the wire and the effect of the bubble "noses" cancel each other out.

Some results have been obtained from experiments conducted at higher film temperatures (Fig. 11). These results show a slight increase in heat-transfer coefficients for the case of higher film temperatures which is probably due to the increased contribution of liquid convection. These results do not differ appreciably from results obtained from experiments conducted under standard conditions ($T_F = 25^{\circ}C$) and confirm once more that transient conduction is the most important mechanism of the bubble induced heat transfer.

The theoretical solution to the present problem shows (and experimental results confirm) that the frequency of bubble generation has profound effect on the time–mean bubble induced heat transfer. The time–mean heat-transfer coefficient increases with the frequency of bubble generation. Hence in order to maximize the time–mean bubble induced heat transfer the frequency of bubble generation should be as high as possible.

6. CONCLUSIONS

Bubble induced heat transfer in simplified gas–liquid systems with controlled frequency of bubble generation, using a special heat-transfer probe, has been investigated. It has been found that:

(i) Surface renewal and penetration theory can be used to describe the bubble induced heat transfer in the present system to a good degree of accuracy.

(ii) Bubble wakes are primarily responsible for the surface renewal of the liquid.

(iii) Transient conduction into the liquid phase is the most important mechanism of the bubble induced heat transfer, being responsible for about 75 per cent of the total heat transfer. Convection contributes the remainder.

(iv) To maximize the time–mean bubble induced heat transfer, the frequency of bubble generation should be as high as possible.

(v) To allow high frequencies of bubble generation without bubble coalescence and to decrease the residence time of bubbles on the wire surface, the volume of generated bubbles should be as small as possible.

(vi) The only possible advantage of larger gas bubbles is that their velocity is greater, thus increasing the contribution of the additional forced convection to the overall bubble induced heat transfer. Nevertheless, the corresponding small increase in heat-transfer rates is far outweighed by the adverse effects associated with large gas bubbles, namely, large bubble residence time on the wire surface and the limit on the maximum frequency of generation of discrete gas bubbles.

Acknowledgements—The author's thanks are due to The University of Aston in Birmingham for supporting this work. Thanks are also due to Professor D. E. Elliott, who supervised this work, and to Mr J. R. Howard of the Department of Mechanical Engineering, University of Aston, for their help, suggestions and criticism during the preparation of this paper.

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APPENDIX 1

Simplification of Equations (3)–(9)

Since the wire is approximated by a finite rod with uniform radial temperature distribution, equations (4) and (9) are not applicable. Energy balance on an element of the wire shows that for

$$t \geqslant 0,\ r = a,\ |z| \leqslant L$$

$$\frac{k_W}{\kappa_W}\frac{\partial T_{W1}}{\partial t} = k_W\frac{\partial^2 T_{W1}}{\partial z^2} - \frac{2}{a}F_s + Q \qquad \text{(A.1)}$$

where F_s is the heat flux which is being dissipated from the outer surface of the wire by conduction into the surrounding liquid and hence

$$F_s = -k_L\left(\frac{\partial T_{L1}}{\partial r}\right)_{r=a}. \qquad \text{(A.2)}$$

Equation (A.2) is substituted into equation (A.1)

$$t \geqslant 0,\ r = a,\ |z| \leqslant L$$

$$\frac{1}{\kappa_W}\frac{\partial T_{W1}}{\partial t} = \frac{\partial^2 T_{W1}}{\partial z^2} + \frac{2k_L}{ak_W}\left(\frac{\partial T_{L1}}{\partial r}\right)_{r=a} + \frac{Q}{k_W} \quad \text{(A.3)}$$

and equation (8) is then substituted into equation (A.3) to obtain for

$$t \geqslant 0,\ r = a,\ |z| \leqslant L$$

$$\frac{1}{\kappa_W}\frac{\partial T_{L1}}{\partial t} = \frac{\partial^2 T_{L1}}{\partial z^2} + \frac{2k_L}{ak_W}\frac{\partial T_{L1}}{\partial r} + \frac{Q}{k_W}. \quad \text{(A.4)}$$

Equation (A.4) is identical to equation (22). Hence the wire and the liquid temperatures are uncoupled and the liquid temperature, T_{L1}, can be obtained from equations (18)–(22).

The wire temperature, T_{W1}, is calculated from equation (8) as

$$T_{W1} = (T_{L1})_{r=a}. \quad \text{(A.5)}$$

APPENDIX 2

Method of Solution of Equations (18)–(22)

First the Laplace Transformation of equations (18)–(22) with respect to time is

$$r \geqslant a,\ |z| \leqslant L$$

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} - q^2\Phi = 0 \quad \text{(A.6)}$$

subject to

$$r \geqslant a,\ |z| = L \quad \Phi = 0 \quad \text{(A.7)}$$

$$r \geqslant a,\ z = 0 \quad \frac{\partial \Phi}{\partial z} = 0 \quad \text{(A.8)}$$

$$r = a,\ |z| \leqslant L \quad \frac{p}{\kappa_W}\Phi = \frac{\partial^2 \Phi}{\partial z^2} + \frac{2k_L}{ak_W}\frac{\partial \Phi}{\partial r} + \frac{Q}{pk_W} \quad \text{(A.9)}$$

$$r \geqslant a,\ |z| \leqslant L \qquad T_{L1} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{pt}\left\{\frac{2Q\kappa_W}{k_W p}\sum_{n=0}^{\infty}\frac{1}{s}\frac{(-1)^n K_0(\beta r)\cos\dfrac{sz}{L}}{\left(p+\kappa_W\dfrac{s^2}{L^2}\right)K_0(\beta a) + \dfrac{2k_L\kappa_W}{ak_W}\beta K_1(\beta a)}\right\}\mathrm{d}p. \quad \text{(A.19)}$$

where Φ is the Laplace Transformation of T_{L1} defined by

$$\Phi(p) = \int_0^\infty e^{-pt}T_{L1}\,\mathrm{d}t \quad \text{(A.10)}$$

and

$$q^2 = p/\kappa_L. \quad \text{(A.11)}$$

Next Φ and Q/pk_W are expanded using a cosine Fourier series

$$\Phi = \sum_{n=0}^{\infty} R_n(r)\cos\frac{sz}{L} \quad \text{(A.12)}$$

$$Q/pk_W = \frac{2Q}{pk_W}\sum_{n=0}^{\infty}\frac{(-1)^n}{s}\cos\frac{sz}{L} \quad \text{(A.13)}$$

where $R_n(r)$ is a function of r only and

$$s = \frac{2n+1}{2}\pi. \quad \text{(A.14)}$$

Equations (A.12) and (A.13) are substituted into equations (A.6)–(A.9), which must be identically satisfied for all values of z.

This is possible only if for

$$r \geqslant a$$

$$\frac{\mathrm{d}^2 R_n}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R_n}{\mathrm{d}r} - (q^2 + s^2/L^2)R_n = 0 \quad \text{(A.15)}$$

subject to

$$r = a$$

$$\frac{p}{\kappa_W}R_n = -\frac{s^2}{L^2}R_n + \frac{2k_L}{ak_W}\frac{\mathrm{d}R_n}{\mathrm{d}r} + \frac{2Q}{pk_W}\frac{(-1)^n}{s}. \quad \text{(A.16)}$$

Equation (A.15) is a modified Bessel equation of the zeroth order. Using the implied condition that the temperature at infinity is finite, the solution of equations (A.15) to (A.16) is then obtained in the following form:

$$r \geqslant a$$

$$R_n(r) = \frac{2Q\kappa_W}{psk_W} \times \frac{(-1)^n K_0(\beta r)}{\left(p+\kappa_W\dfrac{s^2}{L^2}\right)K_0(\beta a) + \dfrac{2k_L\kappa_W}{ak_W}\beta K_1(\beta a)} \quad \text{(A.17)}$$

where

$$\beta^2 = q^2 + s^2/L^2 \quad \text{(A.18)}$$

and $K_0(\beta a)$ and $K_1(\beta a)$ are modified Bessel functions of the second kind.

Substituting (A.17) into (A.12) and using the Inversion Theorem for the Laplace Transformation an expression for T_{L1} is then obtained:

Reversing the order of integration and summation, using the theorem for the Laplace Transformation of an integral and writing

$$p' = p + \kappa_L\frac{s^2}{L^2} \quad \text{(A.20)}$$

$$q'^2 = \beta^2 = p'/\kappa_L \quad \text{(A.21)}$$

equation (A.19) is simplified to

$$T_{L1} = \frac{2Q\kappa_W}{k_W}\sum_{n=0}^{\infty}\frac{(-1)^n}{s}\cos\frac{sz}{L}\int_0^t \exp\left[-\kappa_L\left(\frac{s}{L}\right)^2\lambda\right]W(\lambda)\,\mathrm{d}\lambda \quad \text{(A.22)}$$

where

$$W(\lambda) = \frac{1}{2\pi i}\int_{\sigma' - i\infty}^{\sigma' + i\infty} e^{p'\lambda} \frac{K_0(q'r)}{\left[p' + (\kappa_W - \kappa_L)\dfrac{s^2}{L^2}\right]K_0(q'a) + \dfrac{2k_L\kappa_W}{ak_W}q'K_1(q'a)}\mathrm{d}p'. \quad \text{(A.23)}$$

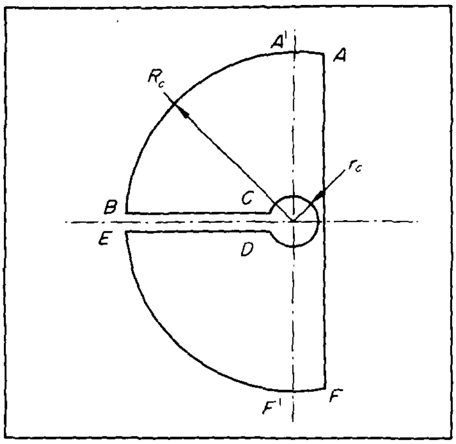

FIG. A1. Contour used for the Inversion Theorem.

Because the integrand of (A.23) has a branch point at the origin, a contour shown in Fig. A.1 is used for evaluating of (A.23). The integrand is then a single valued function inside the closed contour. It is known that if $\kappa_W > \kappa_L$ (as in the case of most liquids) there are no poles of the integrand inside this region and on its boundaries [11]. It is simple to show that integral (A.23) round the small circle tends to zero as $r_c \to 0$ and that this integral along the contours A to B and E to F (Fig. A.1) also tends to zero as $R_c \to \infty$. Thus when $r_c \to 0$ and $R_c \to \infty$, integral (A.23) can be substituted by the sum of real infinite integrals over BC and DE.

To obtain the real infinite equivalent of (A.23), $\kappa_L (x/a)^2 e^{-i\pi}$ and $\kappa_L (x/a)^2 e^{i\pi}$ are substituted for p' [equation (A.20)] on contours DE and BC respectively (Fig. A.1). This is then substituted into equation (A.22). Changing the order of integration and finally integrating with respect to time leads to an expression for the temperature field in the liquid surrounding the wire in the form given by equations (23)–(26).